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M. Sc. [Sem I] [Paper I]

Algebra

Extensions (contd.)

Definition $\longrightarrow$ already taken up.

Types

I. Let E be an extension field of a field F and let $a \in E$. We call 'a' algebraic over F if a is the zero of some non-zero polynomial in $F[x]$.

If a is not algebraic over F, it is called transcendental over F.

An extension E of F is called an algebraic extension of F if every element of E is algebraic over F.

If E is not an algebraic extension of F, it is called a transcendental extension of F.

An extension of F of the form $F(a)$ is called a simple extension of F.

Theorem: CHARACTERIZATION OF EXTENSIONS

Q Let E be an extension field of the field F and let $a \in E$. If a is transcendental over F then $F(a) = F(x)$. If a is algebraic over F then prove that $F(a) \approx F[x]/\langle p(x)\rangle$ where $p(x)$ is a polynomial in $F[x]$ of minimum degree such that $p(a) = 0$. Moreover, $p(x)$ is irreducible over F.

Proof We consider the homomorphism

$\phi : F[x] \to F(a)$ given by $f(x) \to f(a)$.

If a is transcendental over F then kernel of ϕ i.e. $\ker \phi = \{0\}$.

so we extend ϕ to an isomorphism

$\bar{\phi} : F(x) \to F(a)$ by

$\bar{\phi}(f(x)/g(x)) = f(a)/g(a)$.

If a is algebraic over F then $\ker \phi \neq \{0\}$

So, there is a polynomial $p(x)$ in $F[x]$ s.t. $\ker \phi = \langle p(x)\rangle$ and $p(x)$ has minimum degree among all non zero elements of $\ker \phi$. Thus, $p(a) = 0$.

$\because$ $p(x)$ is a polynomial of minimum degree with this property $\Rightarrow$ it is irreducible over F.